Tunneling in frustrated total internal reflection: A comparison of different approaches

Anedio Ranfagni,[1] Paolo Moretti,[2] Ilaria Cacciari,[1] and Maria Angela Vitali[3]
[1]*Istituto di Fisica Applicata "Nello Carrara" del Consiglio Nazionale delle Ricerche, via Madonna del Piano 10, 50019 Sesto Fiorentino, Firenze, Italy*
[2]*Istituto dei Sistemi Complessi del Consiglio Nazionale delle Ricerche, Sezione di Firenze, via Madonna del Piano 10, 50019 Sesto Fiorentino, Firenze, Italy*
[3]*Scuola di Specializzazione in Ottica dell' Università di Firenze, Firenze, Italy*
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Tunneling processes in frustrated total internal reflection are reexamined in order to compare the results already obtained from adopting a procedure based on a transition-element analysis with those recently reported on the basis of a statistical method applied to a Brownian-like motion. A close agreement between the two approaches can be established, at optical and microwave scales, when suitable values of the involved parameters are adopted.

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In a previous paper [1], it was demonstrated that a statistical method based on a path-integral treatment of a Brownian-like motion is capable of interpreting the results of delay-time in frustrated total internal reflection (FTIR) at both optical and microwave scales. We wish to recall, however, that the same results have been already interpreted: Always within the framework of path integrals, but in accordance with a procedure that is based on transition-element analysis [2].

Both approaches adopt path integration, but are completely different [3], although suitable for interpreting the same reality. The purpose of this study is to further investigate this problem, in an attempt to find under what conditions a close agreement between the two approaches can be obtained.

First let us briefly summarize the procedure relative to transition-element analysis. The transition element (a sort of average) of the trajectory $\langle\tilde{x}\rangle$ can be evaluated by differentiating the transition element of ΔS, that is, the variation in the action due to dissipative effects. This is given by $\Delta S = S' - S = \int dt\, f(t)x(t)$, with S being an unperturbed quadratic action, $f(t)$ any arbitrary function, to be identified, in our case, with $\eta\dot{\bar{x}}$, with η being the dissipative constant and $x(t)$ a path. By an expansion of $\exp(i\Delta S/\hbar)$ in power series to first order, $\langle\tilde{x}\rangle \simeq \bar{x}(t)(1+i\Delta S/\hbar)\langle 1\rangle_S$, where $\bar{x}(t)$ is the classical path and $\langle 1\rangle_S$ the propagator.

The equation of motion adopted is similar to that of a damped harmonic oscillator which, in view of the forbidden character of the process, is written as $\ddot{x}(t)+2ia\dot{x}(t)+\omega^2 x(t) = 0$, where the damping parameter $a=\eta/2m$ (m being the mass of the particle and ω the angular frequency), is replaced by ia. With the boundary conditions $x(0)=0$ and $\dot{x}(0)=v$, the classical path is given by

$$\bar{x}(t) = \frac{v}{\tilde{\omega}}\sin(\tilde{\omega}t)e^{-iat}, \tag{1}$$

where $\tilde{\omega}=(\omega^2+a^2)^{1/2}$, and the velocity is given by

$$\dot{\bar{x}} = v\left[\cos(\tilde{\omega}t) - \frac{ia}{\omega}\sin(\tilde{\omega}t)\right]e^{-iat}. \tag{2}$$

The leading term in the variation of the action becomes

$$\Delta S = \frac{imav^2}{\tilde{\omega}^2}\sin^2(\tilde{\omega}t)e^{-2iat}. \tag{3}$$

By using the identification $mc^2/\hbar \to \tilde{\omega}$ and taking Eqs. (1) and (2) in the limit $a\to 0$, the transition element of the trajectory is found to be

$$\langle\tilde{x}\rangle \simeq \left[\frac{\delta S'}{\delta f(t)}\right]_{f\to 0}\left[1 - \frac{a}{\tilde{\omega}}\left(\frac{v}{c}\right)^2\sin^2(\tilde{\omega}t)e^{-2iat}\right]\langle 1\rangle_S, \tag{4}$$

where

$$\left[\frac{\delta S'}{\delta f(t)}\right]_{f\to 0} = \bar{x}(t) - \frac{ia}{\omega^2}\dot{\bar{x}}(t) \tag{5}$$

is the functional derivative of the action. By substituting in Eq. (4), under the assumption that $\sin^2(\tilde{\omega}t)=0.5$ and $\langle 1\rangle_S \simeq 1$, the real part of $\langle\tilde{x}\rangle$ is given by [4]

$$\mathrm{Re}\langle\tilde{x}\rangle \simeq \bar{x}(t)\left[1 - \frac{a}{2\tilde{\omega}}\left(\frac{v}{c}\right)^2\cos(2at)\right] + \frac{a}{2\tilde{\omega}}\left(\frac{v}{c}\right)^2\frac{a}{\omega^2}\dot{\bar{x}}(t)\sin(2at). \tag{6}$$

We adopt the definitions

$$\frac{a}{2\tilde{\omega}}\left(\frac{v}{c}\right)^2 = k, \quad \frac{a}{\omega^2}\dot{\bar{x}}(t) = A\bar{x}(t), \tag{7}$$

where the adimensional quantities k and A are to be considered as parameters (see below for A).

Moreover, by dividing $\bar{x}(t)$ by velocity v, we obtain an expression for the real part of the transition element of the time, which can be interpreted as the real traversal time of the barrier (the air gap between the two prisms), namely [5]

TABLE I. Coefficients and constant B relative to the cubic expression (10) as deduced from the parameter values which fit the experimental data reported in Ref. [1] [in Fig. 1a for the optical case and in Fig. 1b for the microwave case [6]].

Case	$T_M \sin i$	$3D-\psi T_M$	$2D-\psi T_M$	B
(a)	16.90 (μm)	26.70 (μm)	14.60 (μm)	9.39 (fs/μm)
(b)	24.16 (mm)	56.90 (mm)	35.70 (mm)	11.47 (ps/mm)

$$\mathrm{Re}\langle t\rangle \simeq \frac{\bar{x}}{v}\{1 - k[\cos(2at) - A\sin(2at)]\}. \tag{8}$$

By putting $2at=2a\bar{x}/v=y$, Eq. (8), for $y \ll 1$, becomes

$$\mathrm{Re}\langle y\rangle \simeq \frac{y}{2a}\left[(1-k) + \frac{k}{2}y^2 + kAy\right]C, \tag{9}$$

where C is a suitable numerical constant, to be determined in order to put the results on the same numerical scale.

Under the assumption that $k<1$ and $A<0$, Eq. (9) is found to be of the $y-y^2+y^3$ type. This is the main result of the present analysis, since Eq. (9) can be brought back to a similar expression for the traversal time reported in Ref. [1] by adopting a completely different approach. There, in fact, for the traversal time of the barrier we obtained Eq. (20) in [1], which can be rewritten, with T ($\equiv t$ in [1]) as the coordinate normal to the gap, the maximum value (i.e., the width of the gap) of which is T_M,

$$\tau(T) = B\left[T_M \sin i\left(\frac{T}{T_M}\right) - (3D-\psi T_M)\left(\frac{T}{T_M}\right)^2 + (2D-\psi T_M)\left(\frac{T}{T_M}\right)^3\right]. \tag{10}$$

Here, $B=2\rho/[c\sin(2i)]$, with ρ being the refractive index of the prisms and i the incidence angle (be sure to distinguish it from the imaginary unit); D and $\psi=i-\beta_T$ are the displacement and the angular deviation of the beam, respectively. By identifying T/T_M with the adimensional variable y in Eq. (9), the two approaches really lead to similar expressions.

Here, as follows, we shall try to establish a quantitative comparison that takes into account the experimental results in Ref. [1], in accordance with Eq. (10), by using the data of Table I.

In case (a), Eq. (10) can therefore be written, in order to facilitate the comparison with Eq. (9), as

$$\tau(T) = 9.39\left[0.169\left(\frac{T}{T_M}\right) - 0.267\left(\frac{T}{T_M}\right)^2 + 0.146\left(\frac{T}{T_M}\right)^3\right]10^2\ (\mathrm{fs}), \tag{11}$$

while, in case (b), it can be written as

TABLE II. Parameter values relative to Eq. (9), as obtained from the ones in Table I by resolving the system (13).

Case	k	A	y_M	a
(a)	0.767	−0.663	0.725	5.30×10^{-2} (fs)$^{-1}$
(b)	0.754	−0.783	0.982	4.36×10^{-2} (ps)$^{-1}$

$$\tau(T) = 11.47\left[0.242\left(\frac{T}{T_M}\right) - 0.569\left(\frac{T}{T_M}\right)^2 + 0.357\left(\frac{T}{T_M}\right)^3\right]10^2\ (\mathrm{ps}). \tag{12}$$

By putting $T=T_M$ in both expressions, we can verify that Eq. (9) gives the same results for the relative parameter values and variable y_M value reported in Table II, which were obtained by resolving the following system of equations:

$$(1-k)y_M = (T_M \sin i)/C,$$

$$-kAy_M^2 = (3D-\psi T_M)/C,$$

$$ky_M^3 = 2(2D-\psi T_M)/C, \tag{13}$$

when the value of 10^2 is taken for the numerical constant C and quantity a is given by $(2B)^{-1}$. In such a way, once y_M is scaled to the unity, Eqs. (9) and (10) are really coincident.

We can therefore conclude that the two approaches—leading to very similar expressions for the traversal time—both describe the experimental results reasonably well, even if the maximum values of the variable y, which are y_M in Table II, do not satisfy well the condition required for the validity of Eq. (9), namely $y \ll 1$. By using the same parameter values in Eq. (8), rather than in Eq. (9), we obtain curves which appreciably deviate for $y>0.5$, so that one or more extra terms in the expansion of Eq. (8) could be considered. Analogously, even in the approach of Ref. [1], higher order terms in Eq. (10) should be considered, but this overcomes the purposes of the present work based on a comparison of results already obtained.

Evaluation of quantity A. According to the definitions (7), quantity A is given by

$$A = \frac{a}{\omega^2}\frac{\dot{\bar{x}}(t)}{\bar{x}(t)} = a\frac{\tilde{\omega}}{\omega^2}\left[\cot(\tilde{\omega}t) - i\frac{a}{\tilde{\omega}}\right], \quad \tilde{\omega} = \sqrt{\omega^2 + a^2}. \tag{14}$$

Until now, quantity a (which in tunneling cases represents an imaginary dissipation, i.e., $a \to ia$) has been considered a real number, with the dimensions of ω. However, it is plausible to introduce a truly dissipative effect by including a suitable imaginary part, so that $a \to a_c = a - ib, b>0$ [7]. Therefore, under the assumption that $|a_c|/\omega \ll 1$, we can write

$$\tilde{\omega} \simeq \omega + \frac{\alpha}{2\omega} - i\frac{\beta}{2\omega}, \quad \alpha = a^2 - b^2, \quad \beta = 2ab. \tag{15}$$

If we consider the behavior of $\cot(\tilde{\omega}t)$ in the complex plane, it is easy to show that

$$\lim_{t\to+\infty} \cot(\tilde{\omega}t) = \mp i, \tag{16}$$

where the (+) sign holds if $a>0$, and (−) if $a<0$. Therefore, at long times, by choosing $a<0$, we can write

$$A \simeq -\frac{i}{\omega^2}(a_c\tilde{\omega} + a_c^2) \tag{17}$$

from which it follows that

$$\mathrm{Re}\,A \simeq -\frac{b}{\omega}\left(1 + \frac{2a}{\omega} + \frac{3a^2 - b^2}{2\omega^2}\right) < 0. \tag{18}$$

In conclusion, we can see that, choosing $a<0$ (remember that the only constraint on a_c is $b<0$), the result $\mathrm{Re}\,A<0$ is obtained, which is equivalent to the assumption $A<0$ of the text when A is analytically continued for complex values.

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[2] A. Agresti, P. Sandri, C. Ranfagni, A. Ranfagni, and R. Ruggeri, Phys. Rev. E **66**, 067604 (2002).

[3] See, R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), Chap. 7 for the transition elements, and Par. 12-6 for the Brownian motion.

[4] See Eq. (15) in Ref. [2], where the last term is reported with a wrong sign. This however does not invalidate the results there obtained, since this term was then neglected while, in the present analysis, it is maintained.

[5] According to the assumption made in Ref. [2], this means that $\bar{x}$ in Eq. (8) and y in Eq. (9) have to be identified with the thickness of the barrier, while v assumes the meaning of a mean velocity of the order of the light velocity in the gap. This crude assumption was legitimated by the plausibility of the result obtained. See also Ref. [7] for an alternative interpretation of the data relative to the microwave case, based on a stochastic process, which leads to results very similar to those of Refs. [1,2].

[6] We have to note that in the parameter values reported in Fig. 1(b) of Ref. [1] there are some mistakes. A reasonable description of the experimental data for the microwave case is obtained with the following values: T_M=27.9 mm, D=21.2 mm, ψ=14°, i=60°, which give the numerical constants in Table I.

[7] A. Ranfagni, R. Ruggeri, C. Susini, A. Agresti, and P. Sandri, Phys. Rev. E **63**, 025102(R) (2001).